UNIT ONE

Exponent Laws & Polynomials

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Important Term and Terminology

Algebraic Expression: A collection of symbols, including one or more variables and possibly numbers and operation symbols. For example, 3x+6, x, 5x, and 2l – 2w are all algebraic expressions.

Exponent: A special use of a superscript in mathematics. For example, 3^2, the exponent is 2. An exponent is used to denote repeated multiplication. For example, 5^4, means 5x5x5x5.

First-degree equation: An equation in which the variable has the exponent one (1); for example, 5(3x – 1) + 6 = - 20 + 7x + 5.

First-degree polynomial: A polynomial in which the variable has the exponent one (1); for example, 3x – 10.

Integer : Any number within the set {…-4, -3, -2, -1, 0, 1, 2, 3, 4 …}

Inverse Operations: Two operations that "undo" or "reverse" each other. For example, addition and subtraction are inverse operations a + b = c means c – a = b. "Squaring" and "taking the square root" are inverse operations, since, for example, $5^2 = 25$ and the (principal) square root of 25 is 5 in other words $\sqrt{25} = 5$.

Monomial: An algebraic expression with one term. Some examples, $3x^2$, 4x, -7.

Polynomial Expression: An algebraic expression taking the form $a + bx + cx^2 + \ldots$, where a, b, and c are numbers.

Rational Number : Any number than can be expressed as a quotient of two intergers where the divisor is NOT zero for example {…-4, …, $\frac{-3}{2}$, …, $\frac{-1}{12}$, …, 0, 1, … $\frac{54}{23}$, …}

Second-degree polynomial: A polynomial in which the variable in at least one of term has an exponent two (2), and no variable has an exponent greater than two (2). For example, $3x^2 + 10$ or $x^2 - 14x + 48$.

Variable: A symbol used to represent an unspecified number. For example, x and y are variables in the expression x + 2y.

Problem Solving Strategy
The Five-Step Process

LIST:	List all known and unknown variables in your problem.
FORMULA(E):	State any useful formulae that may be of use in your problem.
ALGEBRA:	Is your unknown isolated? If not, use algebra to isolate it.
PLUG-IN	Plug in the known variables into your formula(e).
EVALUATE:	Evaluate the problem and conclude with appropriate units.

<u>Learning Goals in this Unit</u>

By the end of this unit, you will be able to:

- derive, through the investigation and examination of patterns, the exponent rules for multiplying and dividing monomials, and apply these rules in expressions involving one and two variables with positive exponents.

- extend the multiplication rule to derive and understand the power of a power rule, and apply it to simplify expressions involving one and two variables with positive exponents.

- add and subtract polynomials with up to two variables [$(3x^2y + 2xy^2) + (4x^2y - 6xy^2)$]

- multiply a polynomial by a monomial involving the same variable [$2x^2 (3x^2 - 2x + 1)$]

- expand and simplify polynomial expressions involving one variable [$2x(4x + 1)-3x(x+2)$]

- solve first-degree equations, including equations with fractional coefficients

- rearrange formulas involving variables in the first degree, with and without substitution (***Sample problem:*** A circular garden has a circumference of 30 m. Write an expression for the length of a straight path that goes through the centre of this garden.);

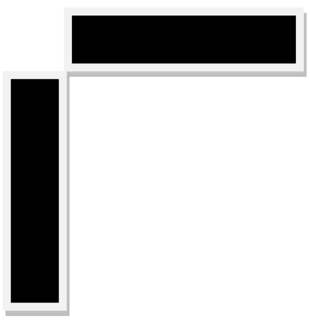
- solve problems that can be modelled with first-degree equations, and compare algebraic methods to other solution methods (***Sample problem:*** Solve the following problem in more than one way: Jonah is involved in a walkathon. His goal is to walk 25 km. He begins at 9:00 a.m. and walks at a steady rate of 4 km/h. How many kilometres does he still have left to walk at 1:15 p.m. if he is to achieve his goal?).

<u>Manipulating Expressions & Solving Equations</u>

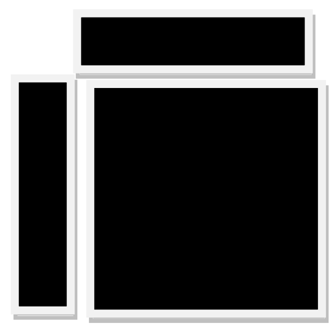
By the end of this unit, you will be able to derive, through the investigation and examination of patterns, the exponent rules for multiplying and dividing monomials, and apply these rules in expressions involving one and two variables with positive exponents.

<u>The Exponent Law of Multiplication</u>

Recall the discussion regarding algeo-tiles. When we review the notion of (x)(x), we see that when they are placed in the form of an area calculation (length)(width), they form a square.

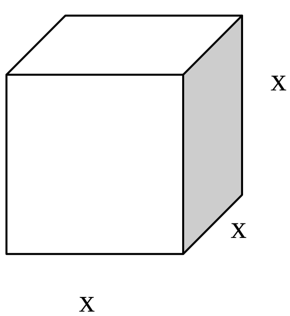


becomes



Hence $(x)(x) = x^2$

If we then multiply $(x^2)(x)$, we then have to lay a square on our desk and then apply an x in an upward direction, thus creating a cube. Note the square on the top with a height x.



Hence $(x^2)(x) = x^3$

Another way of looking at this is to break down the problem into small components. We know that (x^2) is (x)(x); therefore,

$$(x^2)(x) = x^3$$
$$(x)(x)(x) = x^3$$

How many x's are there on the left side?

$$x^3 = x^3$$

Exponent Law of Multiplication

$$(x^a)(x^b) = x^{a+b}$$

What about this one?

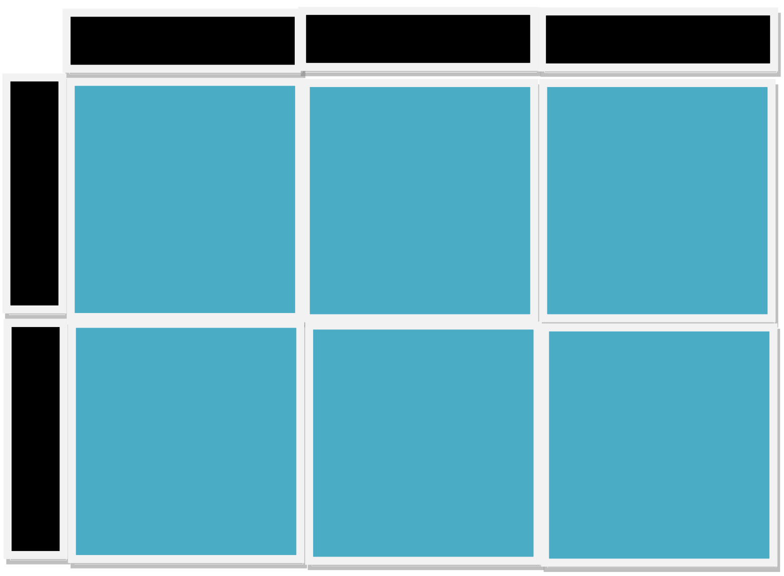
$(2x)(3x) = 6x^2$

Another way to think … $(x^1)(x^1) = x^{1+1}$
or … $(x^1)(x^1) = x^2$

In this case, we are still dealing with (x)(x), which explains the x^2. How do we get the 6? Recall that we are multiplying so we apply the following general rule:

Multiply the coefficients and add the exponents associated with the terms.

Why does it work?



(the row represents 3x's)

How many shapes does it take to fill in the rectangle?

In this case, $6x^2$'s are needed to fill the rectangle.

Blue is used for effect.

(the Column represents 2x's)

This task is aimed at helping you acquire the ability to derive, through the investigation and examination of patterns, the exponent rules for multiplying and dividing monomials, and apply these rules in expressions involving one and two variables with positive exponents.

Recall:

Exponent Law of Multiplication

$$(x^a)(x^b) = x^{a+b}$$

Try these (use any method):

1) $(3x^2)(4x^4) =$ _________ 2) $7(11x^5) =$ _________

3) $(6x^3)(5x) + x =$ _________ 4) $7(11x^5) + 1 =$ _________

Build-up examples: Cover the right side of the page which is reversed.

$(2x)(5) =$ _________	$(2x)(5x^2) = 10x^3$
$(2x)(5x) =$ _________	$(2x)(5x) = 10x^2$
$(2x)(5x^2) =$ _________	$(2x)(5) = 10x$

Why?

Let's review the previous notion that:

$$(x)(x) = x^{1+1}$$
$$(x^1)(x^1) = x^2$$

The first case is simply

$(2x)(5) = (2x^1)(5)$ or 5 2x's.
Hence, 10x or 10x's.

The second case is somewhat as before:

$(2x)(5x) = (2x^1)(5x^1)$ or $10x^{1+1}$.
Hence, $(2x^1)(5x^1) = 10x^2$

The last case will therefore become:

$(2x)(5x^2) = (2x^1)(5x^2)$ or $10x^{1+2}$.
Hence, $(2x^1)(5x^2) = 10x^3$

What do you think would happen with these?

$(2x^2)(5x^2) =$ ________ $(2x^3)(5x^2) =$ ________ $(2x^{21})(5x^{32}) =$ ________

Answers in reverse: $10x^{53}$, $10x^5$, $10x^4$

<u>Exponent Law of Division</u>

Let us analyze a question that involves division. Let's take:

$$\frac{x^5}{x^2}$$

Let's recall that $\frac{x}{x} = 1$. If we were to write the above statement in the following manner:

$$\frac{(x)(x)(x)(x)(x)}{(x)(x)}$$

we would see that there are two ones in the form of $\frac{x}{x}$. We can cross them out and see what we are left with.

$$\frac{\cancel{(x)(x)}(x)(x)(x)}{\cancel{(x)(x)}}$$

What is left?

$$(x)(x)(x)$$

$$x^3$$

Hence, whereas the exponent law of multiplication requires one to add the exponents together, the exponent law of division requires you to subtract the exponents.

<u>Exponent Law of Division</u>

$$\frac{(x^a)}{x^b} = x^{a-b}$$

Hence in the previous example, we could have said:

$$\frac{x^5}{x^2}$$

$$x^{5-2}$$

$$x^3$$

Divide the coefficients and subtract the exponents associated with the terms.

This task is aimed at helping you acquire the ability to derive, through the investigation and examination of patterns, the exponent rules for multiplying and dividing monomials, and apply these rules in expressions involving one and two variables with positive exponents.

Recall:

Exponent Law of Division

$$\frac{(x^a)}{x^b} = x^{a-b}$$

Try these (use any method):

1) $\frac{12x^4}{3x} =$ __________

2) $\frac{20x^3}{4x^2} =$ __________

3) $\frac{45x^5}{5x^2} =$ __________

2) $\frac{49x^3}{7x^3} =$ __________

Build-up examples: Cover the right side of the page which is reversed.

$\frac{64x^5}{16x^4} =$ __________ $4x^5$

$\frac{64x^5}{16x^3} =$ __________ $4x^4$

$\frac{64x^5}{16x^2} =$ __________ $4x^3$

$\frac{64x^5}{16x} =$ __________ $4x^2$

$\frac{64x^5}{16} =$ __________ $4x$

What do you think would happen with these?

$\frac{x^{50}}{x^{10}} =$ __________ $\frac{x^{100}}{x^{25}} =$ __________ $\frac{x^{500}}{x^{100}} =$ __________

Answers in reverse: x^{400}, x^{75}, x^{40}

By the end of this unit, you will be able to extend the multiplication rule to derive and understand the power of a power rule, and apply it to simplify expressions involving one and two variables with positive exponents.

The Power Law

We have seen from the Exponent Law of Multiplication that we add the exponents when multiplying the bases. What happens when we take, say (x^3) to the power of two?

$$(x^3)^2$$

Recall that the square of a number is simply that number times itself. Hence we could say:

$$(x^3)(x^3)$$

Now by the Exponent Law of Multiplication, we can say:

$$x^{3+3}$$

$$x^6$$

Is there a way to bypass the Exponent Law of Multiplication? Note that if we multiplied the exponents, we would have also obtained x^6.

Hence:

The Power Law

$$(x^a)^b = x^{ab}$$

Try these (using the Power Law):

1) $(3x^2)^3$ = _________ 2) $(11x^5)^2$ = _________

3) $(6x^3)^2$ = _________ 4) $(x^5)^5$ = _________

Answers in reverse: x^{25}, $36x^6$, $121x^{10}$, $27x^6$

Build-up examples: Cover the right side of the page which is reversed.

$(2x^2)^2$ = _________ $32x^{10}$

$(2x^2)^3$ = _________ $16x^8$

$(2x^2)^4$ = _________ $8x^6$

$(2x^2)^5$ = _________ $4x^4$

By the end of this unit, you will be able to add and subtract polynomials with up to two variables *[$(3x^2y + 2xy^2) + (4x^2y - 6xy^2)$].*

Recall from that you can only combine LIKE terms given an algebraic expression. In the example below, note that we can only combine the terms with x's with other terms with x's, numbers with numbers and terms with y with other terms with y, etc.

Basic example: Simplify the following expression:

$$4x - 5y + 3 - 2x - 7y - 12$$

As an organizational strategy, one can combine the terms together as a precautionary step:

$$4x - 2x - 5y - 7y + 3 - 12$$

$$2x - 12y - 9$$

However, in this section we will look at more complex polynomials, such as xy, x^2y, xy^2. Note that these terms are all different.

$$4xy - 4x^2y + 7xy^2 + 12xy - x^2y - 9xy^2$$

Consider the terms that are LIKE terms and combine these together.

$$4xy + 12xy - 4x^2y - x^2y + 7xy^2 - 9xy^2$$

$$16xy - 5x^2y - 2xy^2$$

This is as far as we can go without being told the value of x or y.

Now try these:

1) $12x^2y - 4x^2y + 11xy^2 + 12x^2y$ = ________________________________

2) $xz - x^2z - 5xz + 10x^2z$ = ________________________________

3) $(5x^2y - 2xy^2) + (11xy^2 + 12x^2y)$ = ________________________________

4) $(-6y^2z - 4yz^2) + (2y^2z + 12yz^2)$ = ________________________________

TRICK QUESTION) $7xy^2 + 8y^2x$ = ________________________________

Answers in reverse order:

4) $-4y^2z + 8yz^2$ 3) $17x^2y + 9xy^2$ 2) $-4xz + 9x^2z$ 1) $20x^2y + 11xy^2$

TRICK ANSWER) $15xy^2$ OR $15y^2x$:

By the end of this unit, you will be able to multiply a polynomial by a monomial involving the same variable [$2x^2$ ($3x^2$ – 2x + 1)].

Recall the distributive law which states that:

a(b + c) = ab + ac

This law also works for trinomials (an expression with three terms) and so on. Hence,

a(b + c + d) = ab + ac + ad

Basic example: Simplify the following expression:

$4x(x^2 – 5x + 3)$

Note that the 4x outside the bracket gets *distributed* to all the terms inside the bracket through individual multiplication. Don't forget to apply the Exponent Law of Multiplication.

$4x^3 – 20x^2 + 12x$

Ghost example: Recall that –x means (-1)x hence – (x+4) means (-1)(x+4). Expand and then simplify the following expression:

$-3x(x^2 – 2x + 6) – (x^2 + 4x – 5)$

$-3x^3 + 6x^2 – 18x – x^2 – 4x + 5$

Now we combine LIKE terms:

$-3x^3 + 5x^2 – 22x + 5$

Now try these:

1) $–2x(x^2 – 7x + 2)$ = ______________________

2) $4x^2(2x^2 + 5x – 8)$ = ______________________

3) $7x(3x^2 – 2x + 1)$ = ______________________

4) $–x^2(x^3 + 1)$ = ______________________

Answers in reverse order:

4) $– x^5 – x^2$ 3) $21x^3 – 14x^3 + 7x^2$ 2) $8x^4 + 20x^3 – 32x^2$ 1) $–2x^3 + 14x^2 – 4x$

By the end of this unit, you will be able to expand and simplify polynomial expressions involving one variable [2x(4x + 1)-3x(x+2)].

We will again utilize the distributive law and the notion of combining LIKE terms in this section.

Basic example: Simplify the following expression:

$$3x(5x - 7) - 2x(x - 5)$$

We simply use the distributive law twice such that we will then combine LIKE terms where possible.

$$15x^2 - 35 - 2x^2 + 10$$
$$13x^2 - 25$$

Moderate example: Simplify the following expression:

$$5x^2(x - 4) - 7x(x^2 - 1)$$

We will again use the distributive law twice and combine LIKE terms if possible.

$$5x^3 - 20x^2 - 7x^3 + 7x$$
$$-2x^3 - 20x^2 - 7x$$

Challenging example: Simplify the following expression:

$$3x^2(3x^2 + 2x - 6) - 4x(x^2 + 5x - 9) + 4(x^2 + 2x - 3)$$

Here we simply have a larger problem which requires us to use the distributive law thrice and combine LIKE terms if possible.

$$9x^4 + 6x^3 - 18x^2 - 4x^3 - 20x^2 + 36x + 4x^2 + 8x - 12$$

$$9x^4 + 2x^3 - 34x^2 + 44x - 12$$

Now try these:

1) $2x(4x + 1) - 3x(x + 2)$ = ______________________________

2) $3x^2(2x + 3) - 2x(x + 5)$ = ______________________________

3) $4x(6x + 7) - x(2x + 7)$ = ______________________________

Answers in reverse order:

3) $22x^2 + 21x$ 2) $6x^3 + 7x^2 - 10x$ 1) $5x^2 - 4x$

By the end of this unit, you will be able to solve first-degree equations, including equations with fractional coefficients.

Algebra

Recall that in algebra, the ultimate goal is to solve for the unknown variable by **ISOLATION.**

$$\frac{3a}{7} = 6$$

As in previous examples we want to get rid of the multiplying 3 and the dividing 7. We can do this in one step by multiplying **both** sides the reciprocal of the fractional coefficient. That is to say, we can multiply **both** sides by $\frac{7}{3}$.

$$\left(\frac{7}{3}\right)\left(\frac{3a}{7}\right) = \left(\frac{7}{3}\right)(6)$$

Note that on the left side we essentially have created: $\frac{7}{7} = 1$, and $\frac{3}{3} = 1$, thus leaving a isolated.

Hence,

$$a = 14$$

Check:

Left Side	Right Side
$\frac{3a}{7}$	6
$\frac{3(14)}{7}$	
6	

The left side equals the right side, so we are correct; a = 14.

Example:

$$\frac{2x}{5} - 8 = 6$$

Example:

$$\frac{2x}{5} - 8 = 6$$

Strategically we maintain the use of SAMDEB to isolate the variable x. So first we must get rid of the subtracting by adding 8 to both sides.

$$\frac{2x}{5} - 8 + 8 = 6 + 8$$

$$\frac{2x}{5} = 14$$

Now we will multiply both sides by the reciprocal of the fractional coefficient: $\frac{5}{2}$.

$$\left(\frac{5}{2}\right)\left(\frac{2x}{5}\right) = \left(\frac{5}{2}\right)(14)$$

Note that on the left side we have created: $\frac{5}{5} = 1$, and $\frac{2}{2} = 1$, thus leaving x isolated.

Hence,

$$x = 35$$

Check:

Left Side	Right Side
$\frac{2x}{5}$	14
$\frac{2(35)}{5}$	
14	

The left side equals the right side, so we are correct; x = 35.

Now try these:

1) $\frac{6x}{7} = 12$

2) $\frac{2x}{9} - 5 = 1$

Answers in reverse order:

2) x = 27

1) x = 14

In-class task **Name: __________________**

This task is aimed at acquiring the ability to accomplish the first six learning goals in this Algebra Unit.

Simplify the following

1) $(12x)(2x) =$ __________

2) $(-7x)(5x) =$ __________

3) $(6x^2)(3x) =$ __________

4) $2(-13x^2) =$ __________

5) $(5x)(4x^3) =$ __________

6) $(-4x^2)(7x^3) =$ __________

7) $\frac{x^3}{x} =$ __________

8) $\frac{4x^3}{2x^2} =$ __________

9) $\frac{6x^5}{3x} =$ __________

10) $\frac{12x^3}{x^2} =$ __________

11) $(x^2)^3 =$ __________

12) $(2x^3)^3 =$ __________

13) $3(x^3)^2 =$ __________

14) $(-2x^3)^2 =$ __________

15) $(3x^2y - 2xy^2) + (6xy^2 + 7x^2y)$ = ________________________________

16) $(-8x^2y^2 - xy^2) + (9xy^2 - 14x^2y^2$ = ________________________________

17) $-4x(x^2 - 3x + 9)$ = ________________________________

18) $3x(5x^2 + 2x - 1)$ = ________________________________

19) $4x(5x - 1) - 9x(x + 3)$ = ________________________________

20) $2x^2(6x + 5) - x(x + 2)$ = ________________________________

21) $\frac{5x}{3} = 10$

22) $\frac{3x}{7} - 1 = 5$

By the end of this unit, you will be able to rearrange formulas involving variables in the first degree, with and without substitution.

Simple Interest Example

A low risk way to invest money is to lend it to a financial institution. In exchange for lending you the money, the institution will pay you interest. That interest is calculated using a simple interest formula:

$$I = Prt$$

where I is the Interest earned, P is the Principle (the amount that was originally loaned), r is the interest rate and t is the amount of time the money is loaned (usually in years).

If Taylor earned 500.00$ for investing some money for 10 years at 2.35% interest, how much money did he invest?

Solution:

LIST: **I = 500.00$, P = ?, r = 0.0235 and t = 10 years**

Note that r = 0.0235 because 2.35% = 2.35/100 (remember what **per cent** means literally)

FORMULA(E): I = Prt

ALGEBRA: P is not isolated. Algebra is required to isolate it. How do you get rid of a multiplying rt? We need to divide both sides by rt.

$$\frac{Prt}{rt} = \frac{I}{rt}$$

$$P = \frac{I}{rt}$$

PLUG-IN $P = \frac{(500)}{(0.0235)(10)}$

EVALUATE: P ≈ 2127.66$

Taxation problem: Most goods and services in Ontario are subject to a 13% tax under the Harmonized Sales Tax (or HST). The formula is:

$$F = 1.13P$$

where F is the final amount and I is the initial sales price before taxes. Solve for P to understand how much every dollar is really worth.

Solution:

LIST: **F = ?, P = ? We are not asked to find the value of P or F.**

FORMULA(E): $F = 1.13P$

ALGEBRA: $1.13P = F$

$\frac{1.13P}{1.13} = \frac{F}{1.13}$

$P = \frac{F}{1.13}$

PLUG-IN Nothing to plug in.

EVALUATE: $P \approx 0.885F$

This example illustrates that every dollar you possess is actually worth about 89 cents.

Sample problem: A circular garden has a circumference of 30 m. Write an expression for the length of a straight path that goes through the centre of this garden.

Solution:

LIST: **C = 30m , d = ?.**

FORMULA(E): $C = \pi d$

ALGEBRA: $\pi d = C$

$\frac{\pi d}{\pi} = \frac{C}{\pi}$

$d = \frac{C}{\pi}$

PLUG-IN $d = \frac{30}{\pi}$

EVALUATE: $P \approx 9.55$ m

By the end of this unit, you will be able to solve problems that can be modelled with first-degree equations, and compare algebraic methods to other solution methods.

Sample problem: Solve the following problem in more than one way:

Jonah is involved in a walkathon. His goal is to walk 25 km. He begins at 9:00 a.m. and walks at a steady rate of 4 km/h. How many kilometres does he still have left to walk at 1:15 p.m. if he is to achieve his goal?

By the end of this unit, you will be able to solve problems that can be modelled with first-degree equations, and compare algebraic methods to other solution methods.

Sample problem: Solve the following problem in more than one way:

Jonah is involved in a walkathon. His goal is to walk 25 km. He begins at 9:00 a.m. and walks at a steady rate of 4 km/h. How many kilometres does he still have left to walk at 1:15 p.m. if he is to achieve his goal?

<u>Solution#1:</u>

LIST: **v = 4 km/h, d = 25 km, t = ?.**

FORMULA(E): $v = \frac{d}{t}$

This is a physics concept by since velocity is given by units of km/h we know the above formula to be true since it is distance over time.
h
ALGEBRA: To isolate t we need it on top.

$$v = \frac{d}{t}$$

$$vt = \frac{dt}{t}$$

$$vt = d$$

$$\frac{vt}{v} = \frac{d}{v}$$

PLUG-IN $t = \frac{25}{4}$

EVALUATE: t = 6.25 h

This means that if he started at 09h00, he will have completed his walkathon at 15h15 or 3h15 (p.m.). At 1h15 or 13h15, he will be 2hours away. Recall that d = vt

Hence, $d = (\frac{4km}{h})(2h)$

$d = (\frac{4km}{h})(2h)$

d = 8 km

He is therefore 8 km away from his goal.

By the end of this unit, you will be able to solve problems that can be modelled with first-degree equations, and compare algebraic methods to other solution methods.

Sample problem: Solve the following problem in more than one way:

Jonah is involved in a walkathon. His goal is to walk 25 km. He begins at 9:00 a.m. and walks at a steady rate of 4 km/h. How many kilometres does he still have left to walk at 1:15 p.m. if he is to achieve his goal?

<u>Solution#2:</u>

LIST: **v = 4 km/h, d = ?**
t = 13h15 – 09h00?
t = 4.25h

FORMULA(E): $v = \frac{d}{t}$

We could try to find d using algebra.

ALGEBRA: To isolate d we need to get rid of t.

$v = \frac{d}{t}$

$vt = \frac{dt}{t}$

$d = vt$

PLUG-IN $d = (\frac{4km}{h})(4.25)$

EVALUATE: d = 17km

Since he must complete 25km, he is 8 km away from his goal.

<u>In-class task</u> **Name: _________________**

This task is aimed at acquiring the ability to accomplish the last two learning goals in this Algebra Unit.

Find the radius of a circle whose circumference is 56.23 cm using algebraic techniques.

A taxi company charges a 4.50$ initial fee and 1.12 for every kilometre driven. If a client needs wants to go home, 35 km from the airport, how far is the client is the driver's meter reads 30.26$? Solve algebraically using more than one way.